Algebraic perspectives of Persistence
The stability of persistence barcodes

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What is persistent homology?
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[Diagram showing the concept of persistent homology with a δ scale and examples of geometric structures at different scales.]
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Persistent homology is the homology of a filtration

- A filtration is a certain diagram $K : \mathbb{R} \to \text{Top}$
  - $\mathbb{R}$ is the poset category of $(\mathbb{R}, \leq)$
  - A topological space $K_t$ for each $t \in \mathbb{R}$
  - An inclusion map $K_s \hookrightarrow K_t$ for each $s \leq t \in \mathbb{R}$
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- Consider homology with coefficients in a field (often \( \mathbb{Z}_2 \))
  \( H_* : \text{Top} \to \text{Vect} \)
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- Consider homology with coefficients in a field (often $\mathbb{Z}_2$)
  $H_* : \text{Top} \to \text{Vect}$
- Persistent homology is a diagram $M : \mathbb{R} \to \text{Vect}$
  (persistance module)
Homology inference
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Given: finite sample $P \subset \Omega$ of unknown shape $\Omega \subset \mathbb{R}^d$

Problem (Homology inference)

*Determine the homology* $H_\ast(\Omega)$. 

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Idea:

- approximate the shape by a thickening $B_\delta(P)$ covering $\Omega$
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Requires strong assumptions:
Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let $M$ be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

- $B_\delta(P)$ covers $\Omega$, and
- $\delta < \sqrt{3/20} \text{reach}(M)$.

Then $H_\ast(M) \cong H_\ast(B_{2\delta}(P))$. 
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![Diagram of a submanifold with points and a function graph]
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![Graph description](image-url)
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![Diagram](image-url)
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![Graph showing homology reconstruction]

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Homology inference using persistence

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$, $\delta > 0$ be such that

- $B_\delta(P)$ covers $\Omega$, and
- the inclusions $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ preserve homology.

Then $H_\ast(\Omega) \cong \text{im } H_\ast(B_\delta(P) \hookrightarrow B_{2\delta}(P))$. 

![Diagram showing homology persistence](image-url)
Stability
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let \( \| f - g \|_\infty = \delta \). Then there exists a matching between the intervals of the persistence barcodes of \( f \) and \( g \) such that
Stability of persistence barcodes for functions

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Persistence and stability: the big picture

Data

point cloud

\[ P \subset \mathbb{R}^d \]
Persistence and stability: the big picture

Data
\[ \downarrow \]
Geometry

point cloud
\[ \downarrow \]
distance
function

\( P \subset \mathbb{R}^d \)

\( f : \mathbb{R}^d \rightarrow \mathbb{R} \)
Persistence and stability: the big picture

Data \rightarrow\text{Geometry} \rightarrow\text{Topology}

point cloud \downarrow\text{distance} \downarrow\text{sublevel sets}\downarrow\text{topological spaces (filtration)}

\begin{align*}
P & \subset \mathbb{R}^d \\
f : \mathbb{R}^d & \rightarrow \mathbb{R} \\
K & : \mathbb{R} \rightarrow \text{Top}
\end{align*}
Persistence and stability: the big picture

Data → Geometry → Topology → Algebra

point cloud

distance

function

sublevel sets

topological spaces (filtration)

homology

vector spaces (persistence module)

$P \subset \mathbb{R}^d$

$f : \mathbb{R}^d \to \mathbb{R}$

$K : \mathbb{R} \to \text{Top}$

$M : \mathbb{R} \to \text{Vect}$
Persistence and stability: the big picture

Data $P \subset \mathbb{R}^d$

Geometry

Topology

Topology

Algebra

Combinatorics

point cloud

function

sublevel sets

topological spaces (filtration)

homology

vector spaces (persistence module)

structure theorem

intervals (persistence barcode)

$K : \mathbb{R} \to \text{Top}$

$M : \mathbb{R} \to \text{Vect}$

$B : \mathbb{R} \to \text{Mch}$
Interleavings

Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ and $G_t = g^{-1}(-\infty, t]$.
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Interleavings

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Applying homology (functor) preserves commutativity

- persistent homology of $f, g$ yields $\delta$-interleaved persistence modules $\mathbb{R} \to \text{Vect}$
Geometric interleavings
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Geometric interleavings

If two persistence modules are $\delta$-interleaved, then there exists a $\delta$-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
- unmatched intervals have length $\leq 2\delta$. 
Algebraic stability of persistence barcodes


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Algebraic stability of persistence barcodes
Barcodes as diagrams
The matching category

A matching $\sigma : S \leftrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$. 

![Diagram of matching category]
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A matching $\sigma : S \rightrightarrows T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.

Composition of matchings $\sigma : S \rightrightarrows T$ and $\tau : T \rightrightarrows U$:
The matching category

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Composition of matchings $\sigma : S \leftrightarrow T$ and $\tau : T \leftrightarrow U$:

Matchings form a category $\text{Mch}$

- objects: sets
- morphisms: matchings
Barcodes as matching diagrams

We can regard a barcode $B$ as a functor $R \rightarrow \text{Mch}$:
Barcodes as matching diagrams

We can regard a barcode $B$ as a functor $\mathbb{R} \to \text{Mch}$:

- For each real number $t$, let $B_t$ be those intervals of $B$ that contain $t$, and
Barcodes as matching diagrams

We can regard a barcode $B$ as a functor $\mathbf{R} \to \mathbf{Mch}$:

- For each real number $t$, let $B_t$ be those intervals of $B$ that contain $t$, and
- for each $s \leq t$, define the matching $B_s \leftrightarrow B_t$ to be the identity on $B_s \cap B_t$. 

\[ \begin{array}{c|c|c|c|c} 
\delta & 0.1 & 0.2 & 0.4 & 0.8 \\
\hline
\end{array} \]
Stability via functoriality?

\[
\begin{align*}
F_t & \leftrightarrow F_{t+2\delta} \\
G_{t+\delta} & \leftrightarrow G_{t+3\delta}
\end{align*}
\]
Stability via functoriality?

\[ H_*(F_t) \rightarrow H_*(F_{t+2\delta}) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ H_*(G_{t+\delta}) \rightarrow H_*(G_{t+3\delta}) \]
Stability via functoriality?

\[ B(H_*(F_t)) \rightarrow B(H_*(F_{t+2\delta})) \]

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Non-functoriality of the persistence barcode

Theorem (B, Lesnick 2014)

There exists no functor $\text{Vect}^R \to \text{Mch}^R$ sending each persistence module to its barcode.
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Proposition

There exists no functor $\text{Vect} \to \text{Mch}$ sending each vector space of dimension $d$ to a set of cardinality $d$. 
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Proposition
There exists no functor $\text{Vect} \to \text{Mch}$ sending each vector space of dimension $d$ to a set of cardinality $d$.

- Such a functor would necessarily send a linear map of rank $r$ to a matching of cardinality $r$. 
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Proposition
There exists no functor $\text{Vect} \to \text{Mch}$ sending each vector space of dimension $d$ to a set of cardinality $d$.

- Such a functor would necessarily send a linear map of rank $r$ to a matching of cardinality $r$.
- In particular, there is no natural choice of basis for vector spaces.