INDECOMPOSABLES IN MULTI-PARAMETER PERSISTENCE

Representation theory: combinatorial aspects & applications to TDA

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joint work with:

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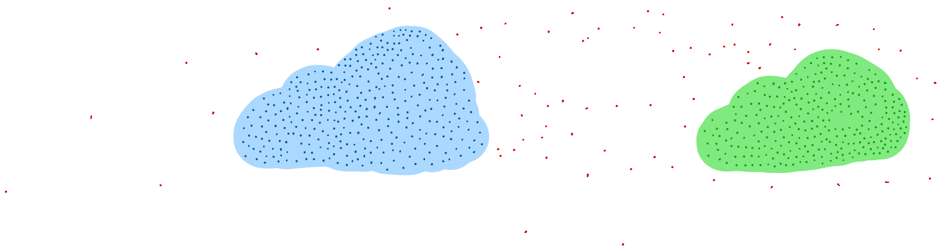
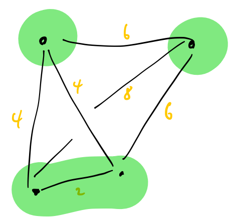


CLUSTERING FUNCTIONS

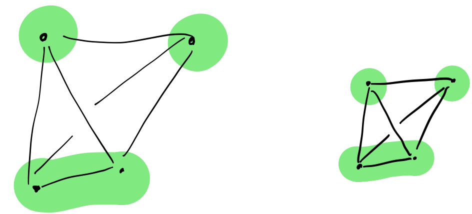
X : finite set

Clustering function φ :
maps a metric $d : X \times X \to \mathbb{R}$ (distance matrix)
to a partition of X.



KLEINBERG'S AXIOMS

Desirable properties



- **scale invariance:**

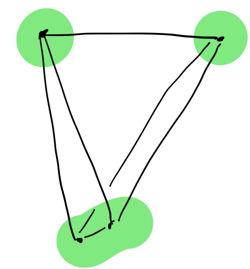
  $$\varphi(d) = \varphi(\lambda \cdot d).$$

- **richness:**

  every partition is obtained from some d.

- **consistency:**

  decreasing d within clusters /
  increasing d across clusters
  does not change the result.



KLEINBERG'S IMPOSSIBILITY THEOREM

Thm [Kleinberg 2002] No clustering function satisfies

- scale-invariant,
- richness, and
- consistency.

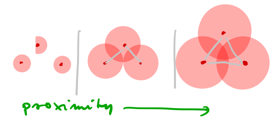
Motivates the use of a scale parameter

⇝ hierarchical clustering

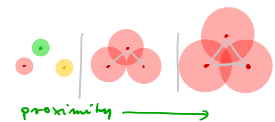
CLUSTERING FROM CONNECTED COMPONENTS

proximity graph

• filter edges by proximity



π_0 (connected components)



single-linkage clustering

H_0 (homology in deg. 0 with coeffs in K)

$$K^3 \to K \to K$$

proximity $\longrightarrow$

persistent homology

$H_0 = F \circ \pi_0$

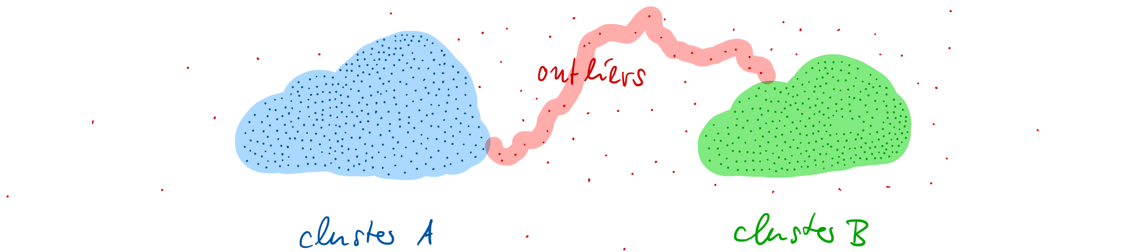
HIERARCHICAL CLUSTERING: EXISTENCE & UNIQUENESS

Thm [Carlsson, Mémoli 2010] Single-linkage clustering is the unique hierarchical clustering method satisfying [... certain axioms similar to Kleinberg's].

But ...

CHAINING EFFECT

Single-linkage clustering is sensitive to outliers

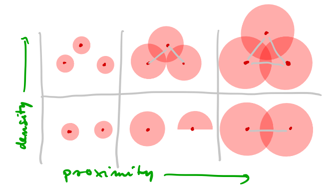


→ not used much in practice!

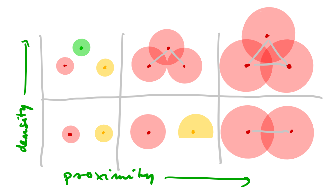
2-PARAMETER CLUSTERING

density-proximity graph

- filter points by density
- filter edges by proximity



π_0 (connected components)



H_0 (homology in deg. 0
with coeffs in K)

$$\begin{tikzcd} K^3 \arrow[r] & K \arrow[r] & K \\ K^2 \arrow[r] \arrow[u] & K^2 \arrow[r] \arrow[u] & K \arrow[u] \end{tikzcd}$$

density

proximity

TRICHOTOMY OF REPRESENTATION TYPES

Given a finite indexing poset P. (k: algebr. closed)

What are the indecomposable diagrams with shape P?

3 cases (representation types): [Drozd 1977]

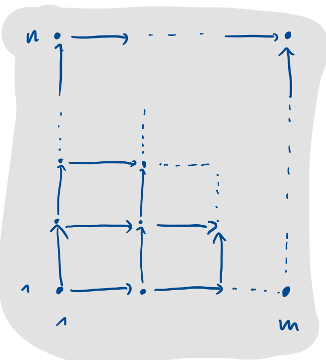
(a) A finite list. **finite type**

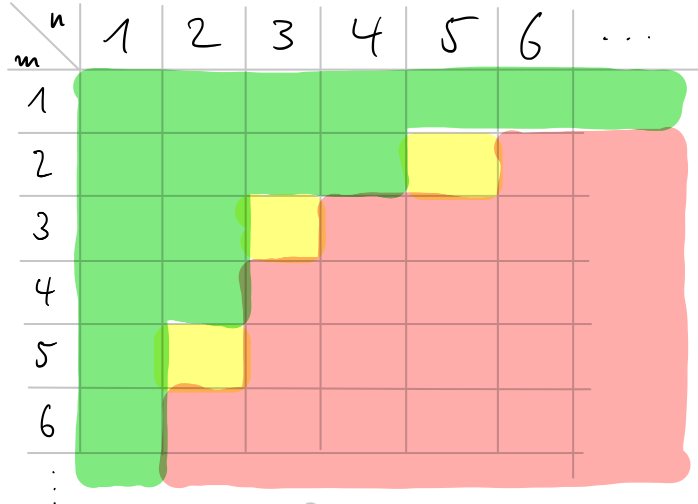
(b) 1-param. families. **tame**

(c) It's complicated. **wild**

(as complicated as modules over any finite-dim. algebra; including undecidable problems)

REPRESENTATION TYPES OF COMMUTATIVE GRIDS





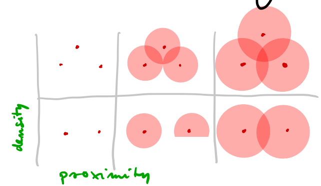
- finite type
- tame
- wild

for $(m-1)(n-1) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 4$.

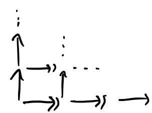
[Leszczyński '94, Skowroński '2000]

GRID DIAGRAMS FROM CLUSTERING

Consider again 2-parameter clustering (proximity / density)



This yields diagrams of the form

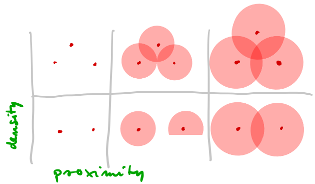


Horizontal maps are surjective!

Does this simplify the picture?

EPIMORPHISMS

Lemma $\mathrm{Rep}^{\twoheadrightarrow}(m,2)$ is finite type.



$$H_0 \rightsquigarrow \begin{tikzcd} K^3 \arrow[r, two heads, "(1\ 1\ 1)"] & K \arrow[r, two heads] & K \\ K^2 \arrow[u, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] \arrow[r, two heads] & K^2 \arrow[u, "(1\ 1)"'] \arrow[r, two heads, "(1\ 1)"'] & K \arrow[u] \end{tikzcd}$$

$$\cong \begin{tikzcd} K \arrow[r, two heads] & K \arrow[r, two heads] & K \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & K \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd}$$

- $\mathrm{Rep}(m,n)$: all commutative dgms over $m \times n$ grid
- $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$: epis in horizontal direction
- $\mathrm{Rep}^{\twoheaduparrow\twoheadrightarrow}(m,n)$: epis in both directions.

EPIC GRIDS & WILD THINGS

Thm [B, Botnan, Oppermann, Steen 20]

$$\begin{tikzcd}
\operatorname{Rep}^{\overset{\twoheadrightarrow}{\twoheadrightarrow}}(m,n) \arrow[r, phantom, "\sim"] \arrow[d, phantom, "\wr"] & \operatorname{Rep}^{\uparrow}(m,n-1) \arrow[d, phantom, "\wr"] \\
\operatorname{Rep}^{\twoheadrightarrow}(m-1,n) \arrow[r, phantom, "\sim"] & \operatorname{Rep}(m-1,n-1)
\end{tikzcd}$$

same representation type

Corollary $\operatorname{Rep}^{\twoheadrightarrow}(m,n)$ is

- finite type
- tame
- wild

for $(m-1)(n-2) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 4$.

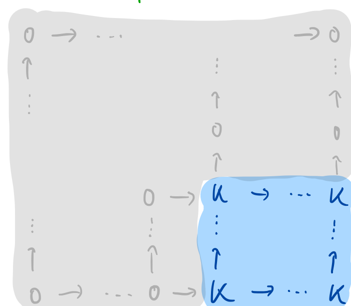
BEHIND THE SCENES

equivalence of categories

$$\frac{\mathrm{Rep}^{\twoheadrightarrow}(m,n)}{\mathrm{Rep}^{\hat{\imath}\twoheadrightarrow}(m,n)} \simeq \mathrm{Rep}(m,n-1)$$

additive quotient $\frac{A}{B}$:
identify morphisms in A whose difference factors through B

indecomposables are of the form



$\Rightarrow$ finite type

The Instability of Decompositions

How useful are indecomposables for TDA?

Thm [B, Scoccola 22] For $n > 1$, among the finitely presented n-parameter persistence modules the indecomposables are dense in interleaving distance.

For every $\varepsilon > 0$, the ε-indecomposables

$$A \oplus B$$

indecomposable $\qquad\qquad d_I(B, 0) < \varepsilon$

form an open & dense subset.

- indecomposability is a generic property [B, Gusel, Scoccola > 24]

THE INSTABILITY OF DECOMPOSITIONS

Proof.

(a) $\forall \varepsilon$, $M : \mathbb{R}^n \to \mathrm{vect}$ f.p., ε-indecomposable

$\exists \delta$:

$d_I(M,N) < \delta \implies N$ is ε-indecomposable.

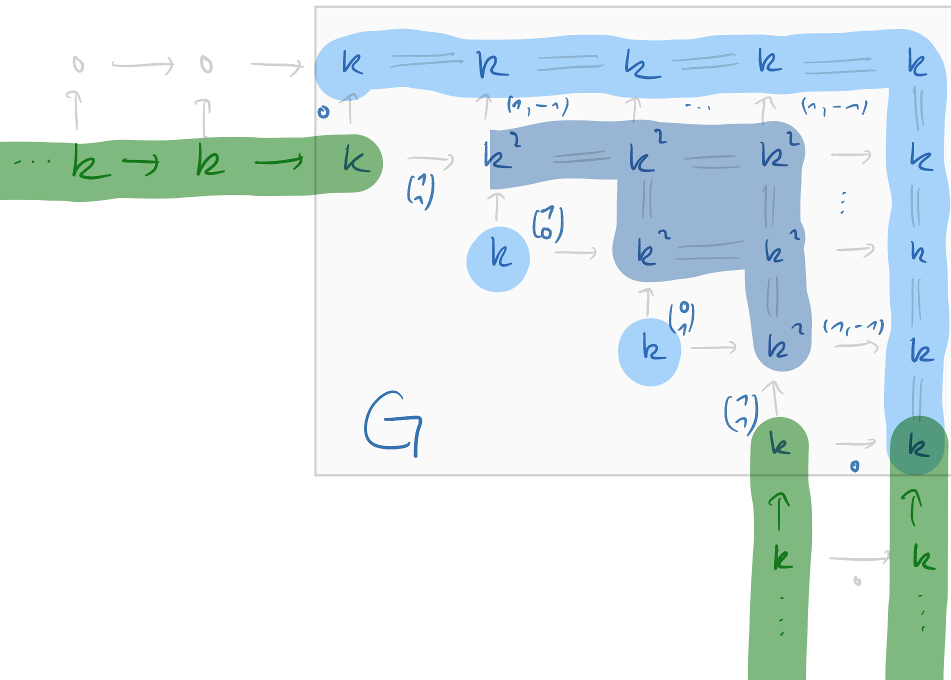
(b) Indecomposables can be "tacked together" with an arbitrarily small change in interleaving distance.

The Idea of Tacking Indecomposables

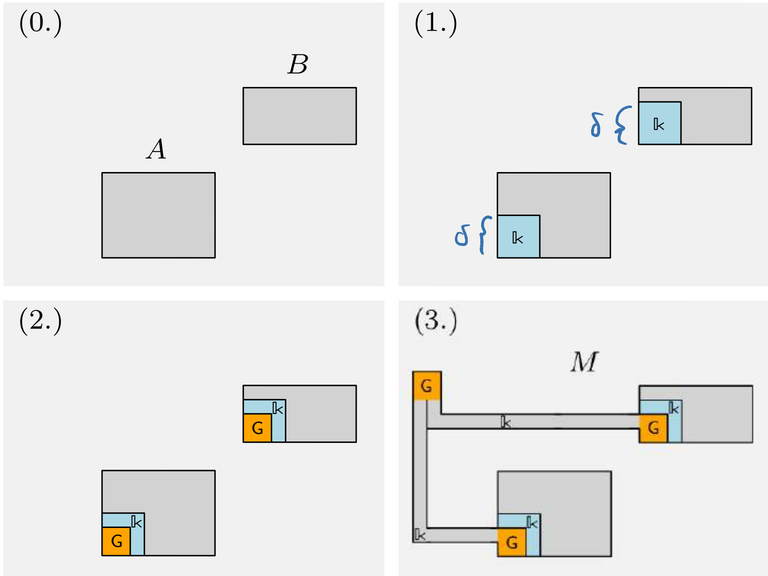
$$
\begin{tikzcd}
k \arrow[r] & k \arrow[r] & k \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\quad \oplus \quad
\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & k \\
0 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\quad \oplus \quad
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & k \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
$$

$$
=
\quad
\begin{tikzcd}
k \arrow[r] & k^2 \arrow[r] & k^3 \\
0 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k^2 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
\quad \twoheadrightarrow \quad
\begin{tikzcd}
k \arrow[r, "\binom{1}{0}"] & k^2 \arrow[r] & k^2 \\
0 \arrow[r] \arrow[u] & k \arrow[r, "\binom{0}{1}"] \arrow[u] & k^2 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u, "\binom{1}{1}"']
\end{tikzcd}
$$

THE TACKING GADGET



TACKING INDECOMPOSABLES



$d_I(A \oplus B, M) < \delta$ for $\delta > 0$ arbitrary

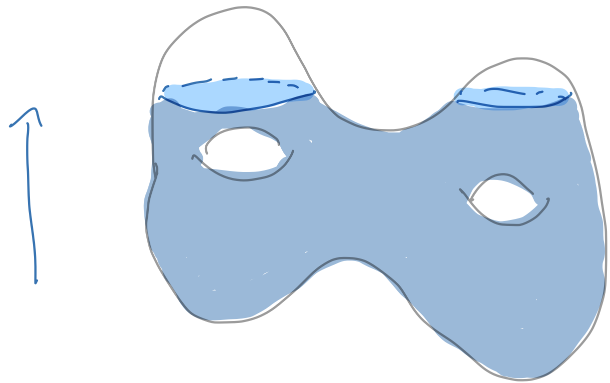
THIN-DECOMPOSABLES ARE NOWHERE DENSE

Can we work with classes of simpler indecomposables?
(e.g. thin: pointwise dim ≤ 1)

Theorem [B, Scoccola '23] Let $\mathcal{F}$ be a class of indecomposable $(n \geq 2)$-parameter persistence modules with pointwise dimension bounded by some constant. Then the $\mathcal{F}$-decomposable persistence modules are nowhere dense.

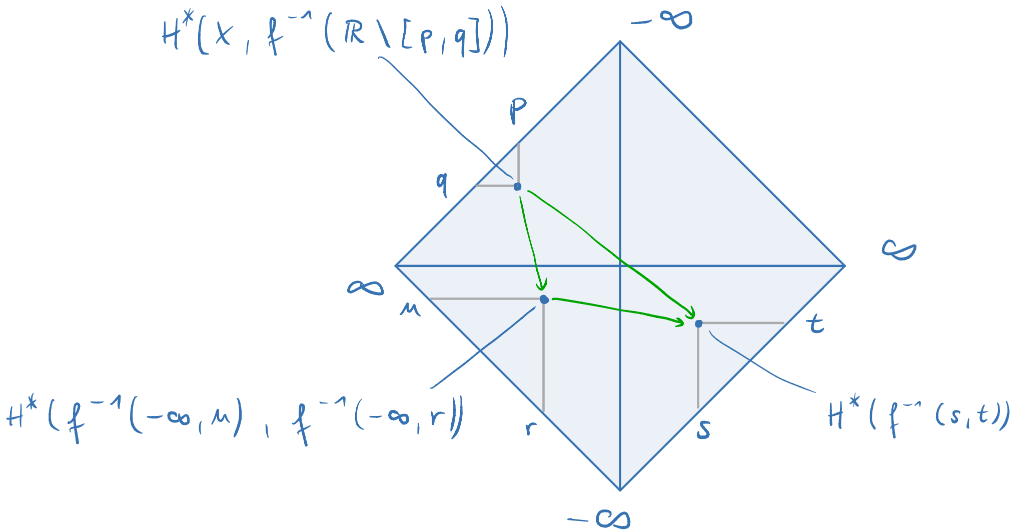
Morse Theory Extended

$f: X \to \mathbb{R}$



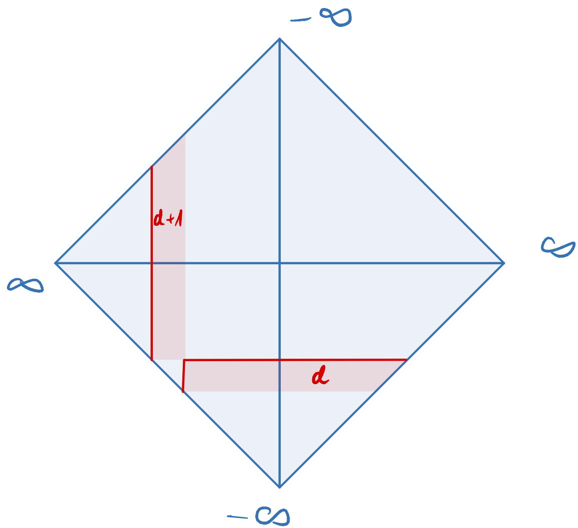
The Mayer-Vietoris Pyramid

[Carlsson et al. 2009] $f : X \to \mathbb{R}$



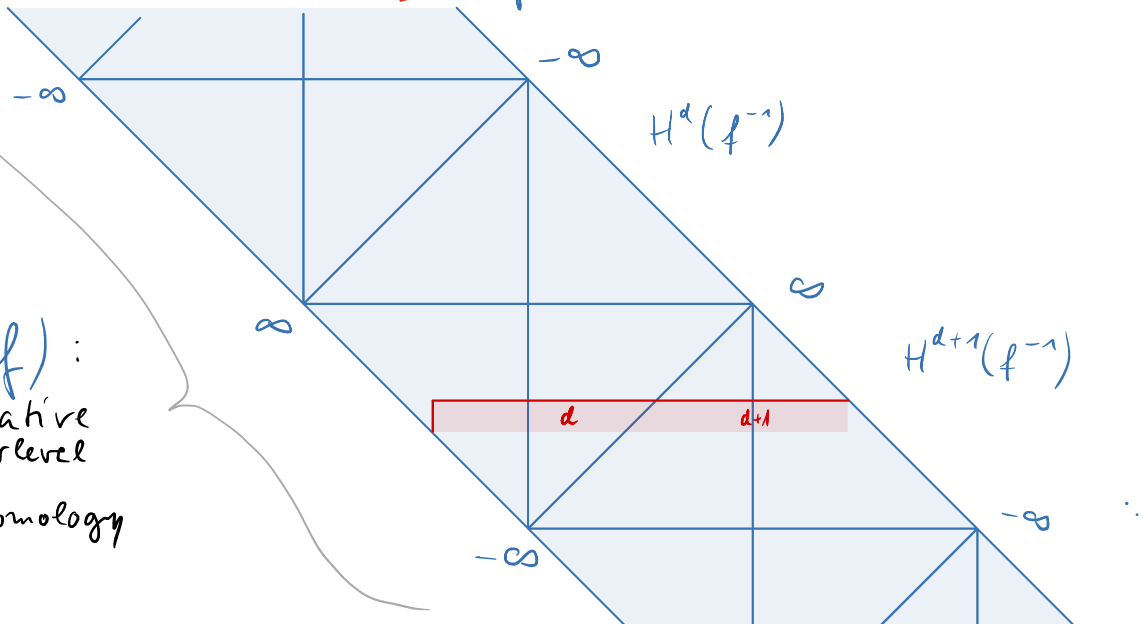
THE MAYER-VIETORIS PYRAMID

[Carlsson et al. 2009] $f : X \to \mathbb{R}$



The Mayer-Vietoris Pyramid

[Carlsson et al. 2009] $f : X \to \mathbb{R}$



$h(f)$:
Relative
Interlevel
Set
Cohomology

RISC OF A FUNCTION

[B, Botnan, Fluhr 2021] Assume $h(f)$ p.f.d.

Prop. $h(f)$ is a 2-parameter persistence module supported on a strip $M \subseteq \mathbb{R}^2$.

Prop. $h(f)$ is cohomological
(equivalently: middle-exact:

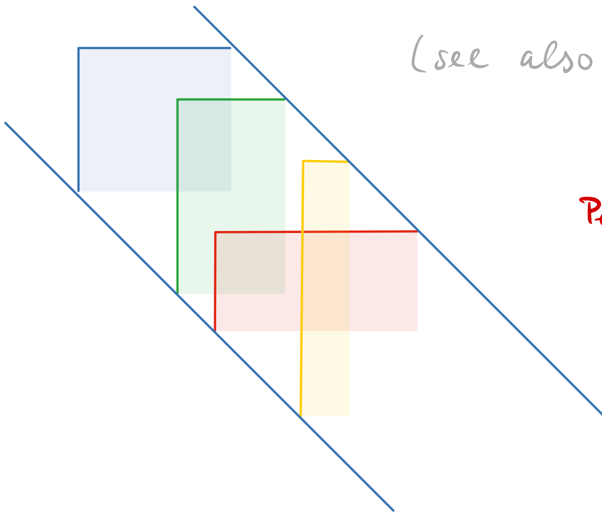
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \rightsquigarrow \quad A \to B \oplus C \to D \text{ exact})$$

Prop. $h(f)$ is sequentially continuous
(for sequences moving up/left in M)

DECOMPOSITION OF COHOMOLOGICAL FUNCTORS

Theorem [BBF '21] Any cohomological sequentially continuous p.f.d. functor $\mathbb{M} \to \mathrm{vect}$ decomposes into rectangle summands.

(see also: [Botnan, Lebovici, Oudot '20])



Proposition $\mathbb{M} \to \mathrm{vect}$ q-tame & sequentially continuous $\Rightarrow$ p.f.d.

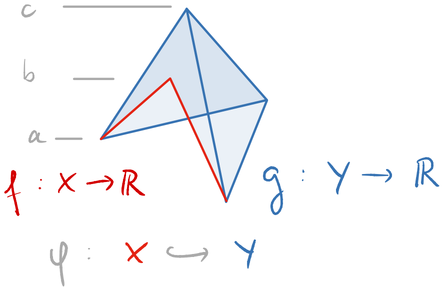
INDUCED MORPHISMS & INTERLEAVINGS

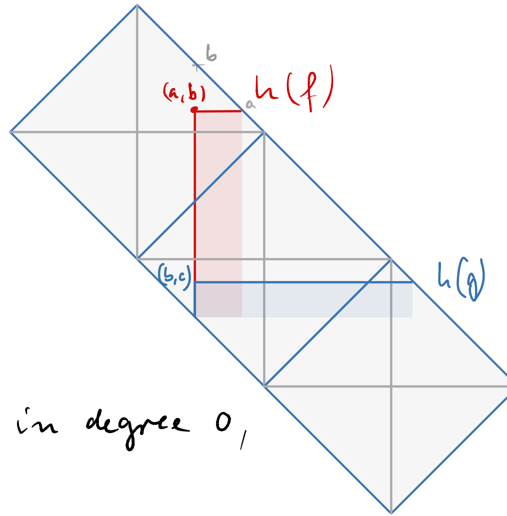
- A map $\varphi :$
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$
induces a morphism in RISC $h(\varphi) : h(f) \to h(g)$.

- Two functions $f, g : X \to \mathbb{R}$ with $\|f - g\|_\infty = \delta$ induce a δ-interleaving between $h(f)$ and $h(g)$.

- These induced morphisms are richer than those in standard persistent homology!

WHAT STANDARD PERSISTENCE CAN'T SEE (BUT RISC CAN)





- f has persistent reduced homology only in degree 0, g only in degree 1.
- Hence the induced map is zero.

But $h(\varphi)$ is nonzero!

EXTENDED & LEVEL SET PERSISTENCE

Extended persistence [Cohen-Steiner, Edelsbrunner, Harer '09]

$$\cdots \hookrightarrow H_*(f^{-1}(-\infty, s]) \hookrightarrow \cdots \hookrightarrow H_*(f^{-1}(-\infty, t]) \hookrightarrow \cdots \hookrightarrow H_*(X) \rightarrow$$

$$\cdots \hookrightarrow H_*(X, f^{-1}[v, \infty)) \hookrightarrow \cdots \hookrightarrow H_*(X, f^{-1}[u, \infty)) \hookrightarrow \cdots$$

Theorem [Carlsson, de Silva, Morozov '09] The persistence diagrams of extended pesistence and level set persistence are in bijective correspondence.

- Some features appear in different degrees across this correspondence

FUNCTORIAL EQUIVALENCE

Can we make the correspondence of extended and level set persistence functorial (an equivalence of categories)?

- Does the correspondence extend to morphisms of persistence modules in a natural way?

Not in the standard way

(persistent homology as graded persistence modules)

Yes using RISC

(level set persistence in a derived category)

RISC $\cong$ DERIVED LEVEL SET PERSISTENCE

Derived
Level
Set
Persistence

Relative
Interlevel
Set
Cohomology

(X locally contractible)

$$\begin{tikzcd}
 & f : X \to \mathbb{R} \ \text{(tame)} \arrow[dl, "Rf_*(k_X):"'] \arrow[dr, "h(f):"] \arrow[d, phantom, "\cong"] & \\
D^+(Sh(\mathbb{R})) \arrow[rr, "\cong"'] & & \mathbb{M} \to \mathrm{vect}
\end{tikzcd}$$

(tame) (cohomological, bounded above)

Theorem [B, Fluhr '22] DLSP $\cong$ RISC.

Depends crucially on morphisms across degrees!

TAKE-HOME MESSAGES

Indecomposables in multi-parameter persistence:

- can be complicated, even in H_0
- are close to every persistence module
- clarify the structure of interlevel set persistence
- illuminate the role of derived categories in persistence