Apparent pairs in computational topology

Ulrich Bauer

Technical University of Munich (TUM)

July 23, 2024

Computational Topology: Foundations, Algorithms, and Applications

AMS-UMI International Joint Meeting, Università degli Studi di Palermo

Funded by

DFG Deutsche Forschungsgemeinschaft German Research Foundation

SFB TRR 109 Discretization in Geometry and Dynamics

Technical University of Munich

MDSI

Munich Center for Machine Learning

In memoriam

Eliyahu Rips

December 12, 1948 – July 19, 2024

Subject: Re: First appearance of the "Rips complex" in your work
Date: Fri, 26 Feb 2021 16:15:00 +0200
From: Eliyahu Rips <eliyahu.rips@mail.huji.ac.il>
To: Fabian Roll <fabian.roll@tum.de>

Dear Prof' Roll,

The story is as follows: Prof. Gromov visited Israel, and I told him some non-published results. He published them (in my name) in his paper on hyperbolic groups. This is the origin of the so-called "Rips complex". In fact, such a complex was earlier discovered by Vietoris (in a somewhat different context).

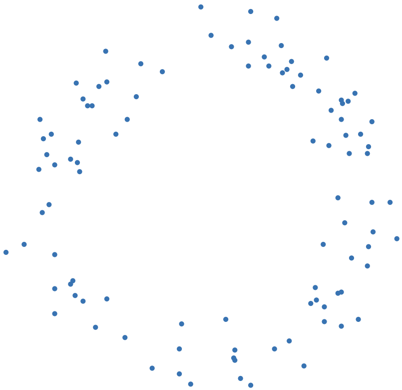
With my best regards,

Eliyahu Rips

Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

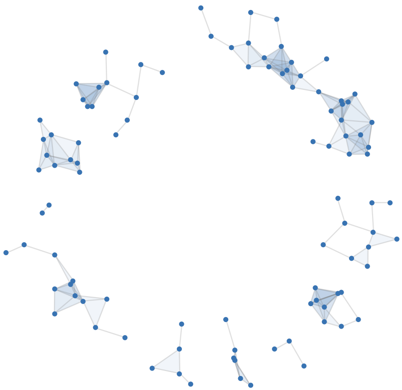
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

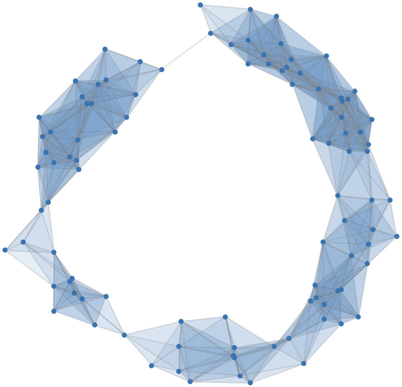
$$\operatorname{Rips}_t(X) = \{S \subseteq X \mid S \neq \emptyset \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

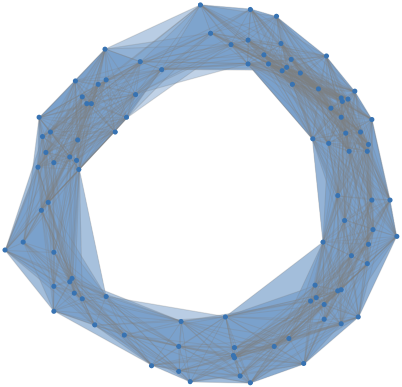
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

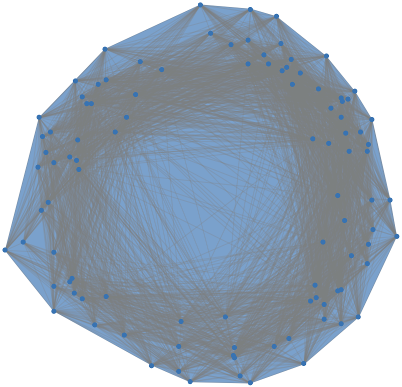
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

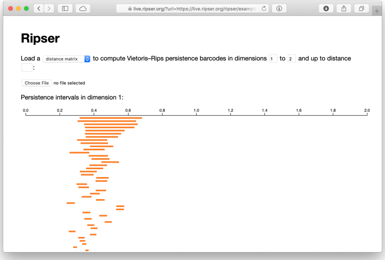
For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

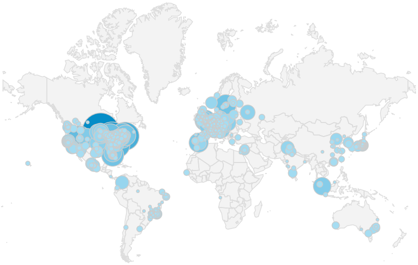
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)

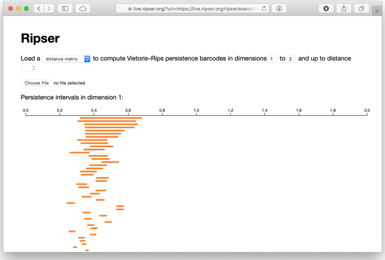


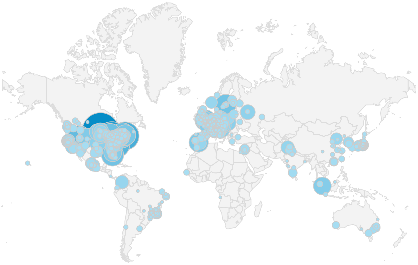


Ripser users worldwide

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





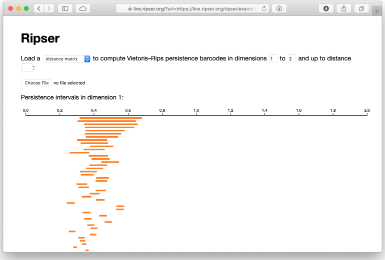
Ripser users worldwide

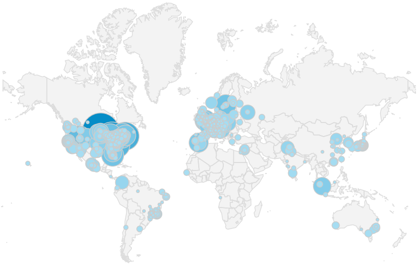
Efficient matrix algorithm based on

- *clearing*: avoiding unnecessary column operations
- computing persistent *cohomology*

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





Ripser users worldwide

Efficient matrix algorithm based on

- *clearing*: avoiding unnecessary column operations
- computing persistent *cohomology*

Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Proposition (B 2021)

The apparent pairs are both

- *persistence pairs (creating/destroying a feature in homology)and*
- *gradient pairs (in the sense of discrete Morse theory).*

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

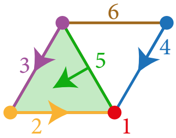
- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Proposition (B 2021)

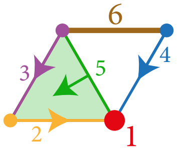
The apparent pairs are both

- *persistence pairs (creating/destroying a feature in homology)and*
- *gradient pairs (in the sense of discrete Morse theory).*

Discrete Morse theory



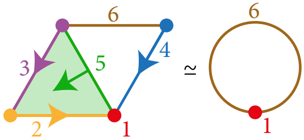
Discrete Morse theory



Discrete Morse theory

Theorem (Forman 1998)

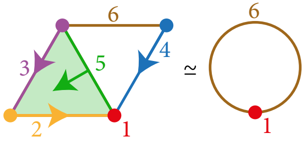
A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



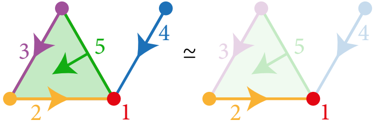
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



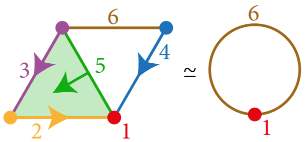
Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



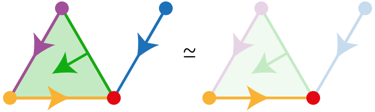
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*

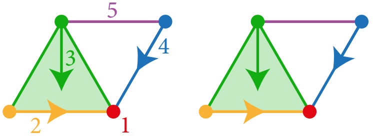


Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



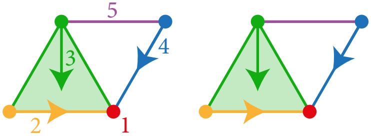
Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

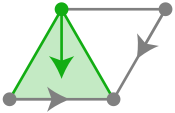


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

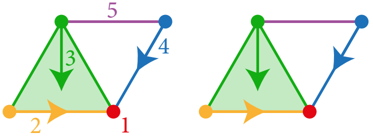


- A generalized vector field V can always be refined to a vector field.

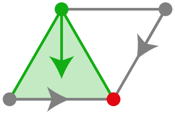


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

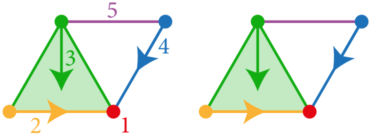


- A generalized vector field V can always be refined to a vector field.

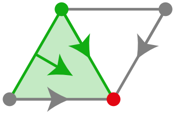


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):



- A generalized vector field V can always be refined to a vector field.



Lexicographically refined Morse filtrations

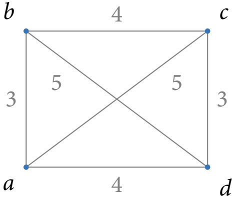
Any generalized discrete Morse function is refined by apparent pairs:

Proposition (B, Roll 2022)

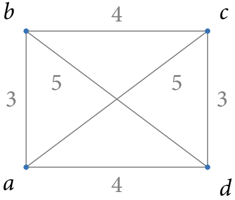
Let f be a generalized discrete Morse function, and consider the simplexwise filtration by lexicographic refinement. Then the apparent pairs of zero persistence form a gradient that

- *refines the gradient of f and*
- *has the same critical simplices.*

Apparent pairs of the diameter-lexicographic filtration



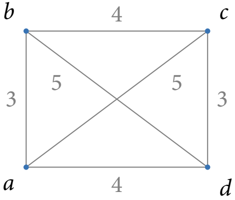
Apparent pairs of the diameter-lexicographic filtration



$$
\partial_1 = \begin{array}{c} \\ \end{array}
\begin{array}{cccccc}
(a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5
\end{array}
$$

$$
\partial_1 = \begin{pmatrix}
1 & & 1 & & 1 & \\
\mathbf{1} & & & 1 & & 1 \\
 & 1 & & 1 & 1 & \\
 & \mathbf{1} & 1 & & & 1
\end{pmatrix}
\begin{matrix} (a) \\ (b) \\ (c) \\ (d) \end{matrix}
$$

Apparent pairs of the diameter-lexicographic filtration



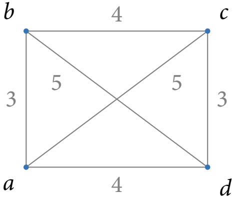
$\partial_1 =$

	(a,b):3	(c,d):3	(a,d):4	(b,c):4	(a,c):5	(b,d):5
(a)	1		1		1	
(b)	**1**			1		1
(c)		1		1	1	
(d)		**1**	1			1

$\partial_2 =$

	(a,b,c):5	(a,b,d):5	(a,c,d):5	(b,c,d):5
(a,b):3	1	1		
(c,d):3			1	1
(a,d):4		1	1	
(b,c):4	1			1
(a,c):5	**1**		1	
(b,d):5		**1**		1

Apparent pairs of the diameter-lexicographic filtration



$$\partial_1 = \begin{array}{c|cccccc|l} & (a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5 & \\ \hline & 1 & & 1 & & 1 & & (a) \\ & \mathbf{1} & & & 1 & & 1 & (b) \\ & & 1 & & 1 & 1 & & (c) \\ & & \mathbf{1} & 1 & & & 1 & (d) \end{array}$$

$$\partial_2 = \begin{array}{c|cccc|l} & (a,b,c){:}5 & (a,b,d){:}5 & (a,c,d){:}5 & (b,c,d){:}5 & \\ \hline & 1 & 1 & & & (a,b){:}3 \\ & & & 1 & 1 & (c,d){:}3 \\ & & 1 & 1 & & (a,d){:}4 \\ & 1 & & & 1 & (b,c){:}4 \\ & \mathbf{1} & & 1 & & (a,c){:}5 \\ & & \mathbf{1} & & 1 & (b,d){:}5 \end{array}$$

$$\partial_3 = \begin{array}{c|c|l} & (a,b,c,d) & \\ \hline & 1 & (a,b,c) \\ & 1 & (a,b,d) \\ & 1 & (a,c,d) \\ & \mathbf{1} & (b,c,d) \end{array}$$

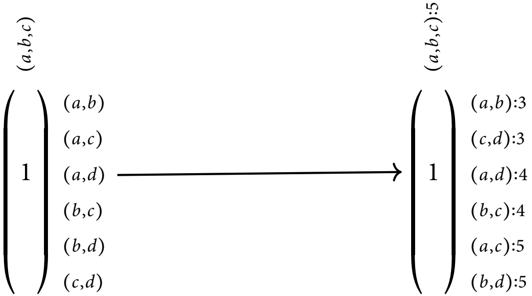
A shortcut for finding pivots

	(a,b,c)
(a,b)	
(a,c)	
(a,d)	
(b,c)	
(b,d)	
(c,d)	

	(a,b,c):5
(a,b):3	
(c,d):3	
(a,d):4	
(b,c):4	
(a,c):5	
(b,d):5	

Shortcut for finding the pivot (latest) facet of a simplex τ:

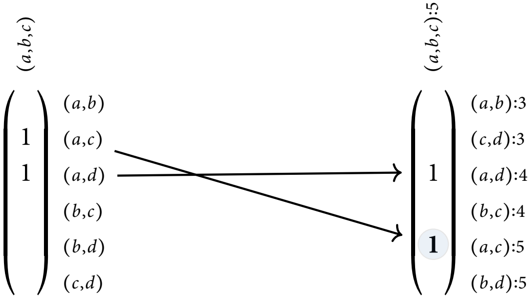
A shortcut for finding pivots



Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order

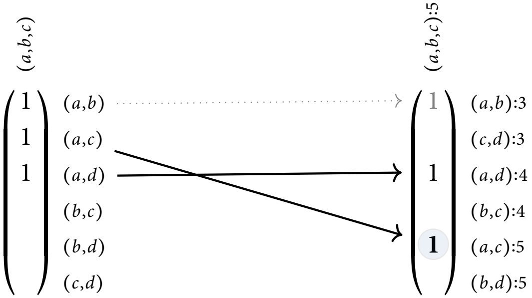
A shortcut for finding pivots



Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam} \sigma = \operatorname{diam} \tau$ is the pivot of τ

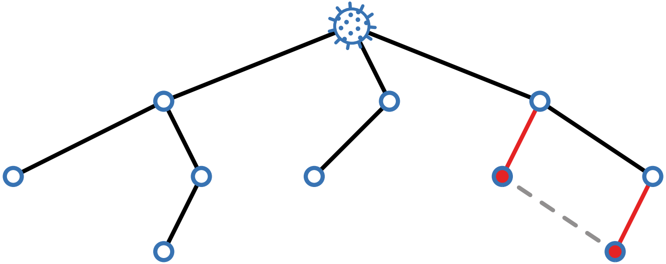
A shortcut for finding pivots



Shortcut for finding the pivot (latest) facet of a simplex τ:

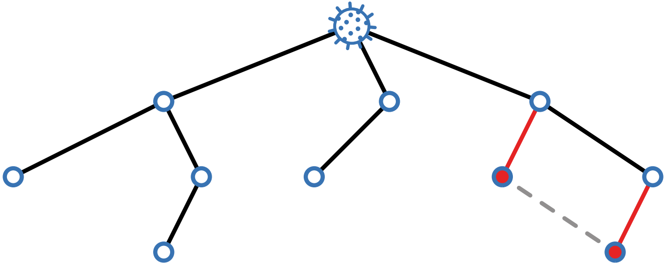
- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam} \sigma = \operatorname{diam} \tau$ is the pivot of τ

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Observation: Ripser runs unusually fast on genetic distance data

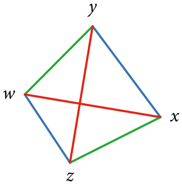
- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- $120\,s$ computation time (with data points ordered appropriately)

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

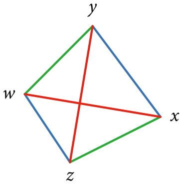


Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



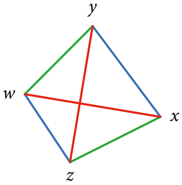
- The hyperbolic plane is $(\ln 2)$-hyperbolic.

Gromov-hyperbolicity

Definition (Gromov 1988)

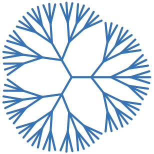
A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



- The hyperbolic plane is $(\ln 2)$-hyperbolic.

- The 0-hyperbolic spaces are precisely the metric trees and their subspaces.



Rips Contractibility

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Rips Contractibility

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

What about

- non-geodesic spaces? In particular, finite metric spaces?
- collapsiblility?
- the filtration?
- the connection to computation of persistent homology?

Rips Contractibility

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

What about

- non-geodesic spaces? In particular, finite metric spaces?
- collapsiblility?
- the filtration?
- the connection to computation of persistent homology?

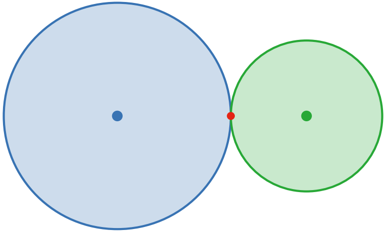
Theorem (B, Roll 2022)

Let X be a finite δ-hyperbolic space. Then there is a single discrete gradient encoding the collapses

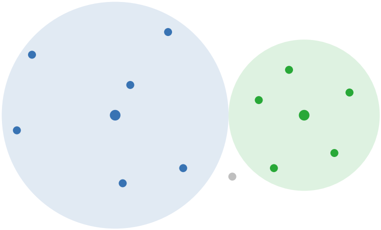
$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

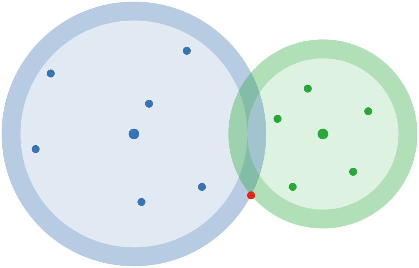
Geodesic defect



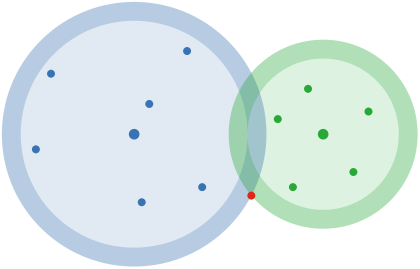
Geodesic defect



Geodesic defect



Geodesic defect



Definition (Bonk, Schramm 2000)

A metric space X is ν-*geodesic* if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \emptyset.$$

The infimum of all such ν is the *geodesic defect* of X.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\mathrm{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The apparent pairs refine this gradient.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances).* *Then the diameter function* $\operatorname{diam}: \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The apparent pairs refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$*,*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V,E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The apparent pairs refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$*,*
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$*, and*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The apparent pairs refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$, *and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval* $(t, u]$.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The apparent pairs refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$, *and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval* $(t, u]$.

In particular, the persistent homology is trivial in degrees > 0.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

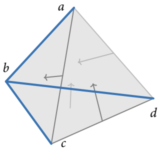
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



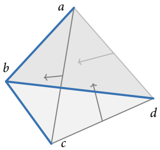
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



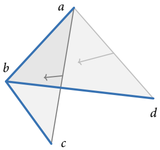
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



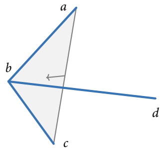
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



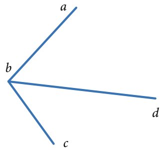
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions. Both functions have the same critical simplices/values.

Morse theory for Čech and Delaunay complexes

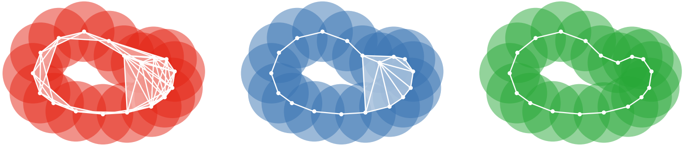
Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions. Both functions have the same critical simplices/values.

Theorem (B, Edelsbrunner 2017)

Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



From Delaunay to Wrap complexes

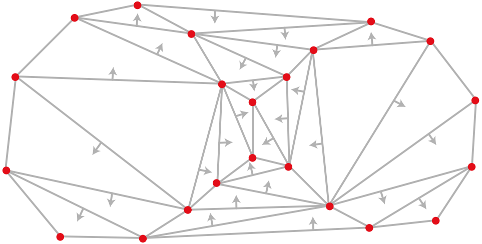
From Delaunay to Wrap complexes

From Delaunay to Wrap complexes

Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)

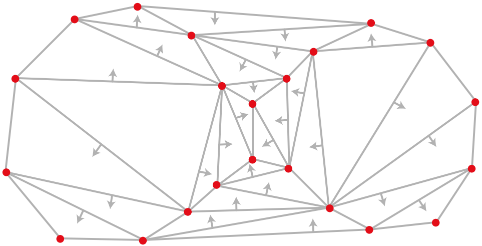
Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



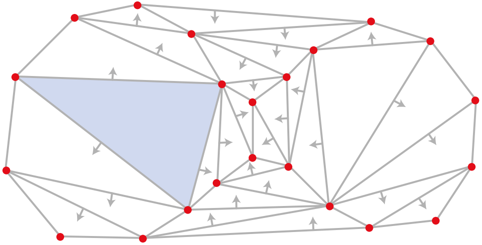
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



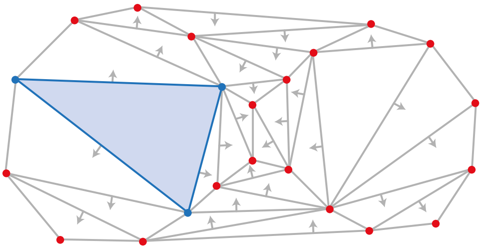
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



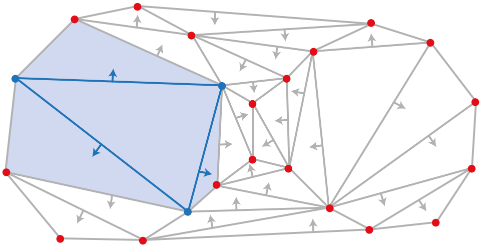
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



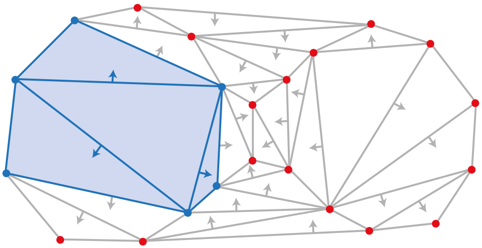
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
    - contains all critical simplices
    - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



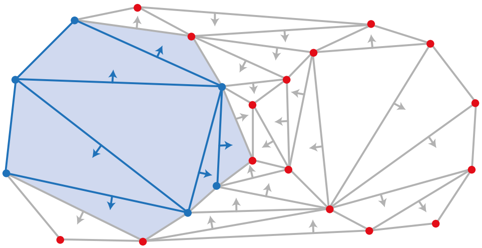
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



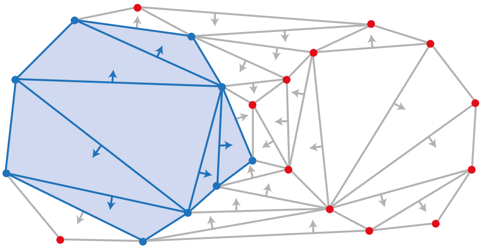
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



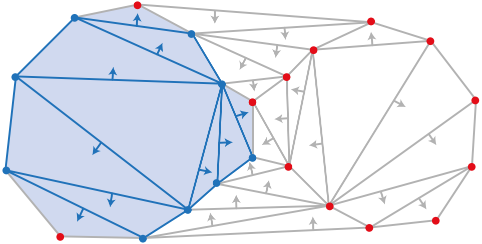
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



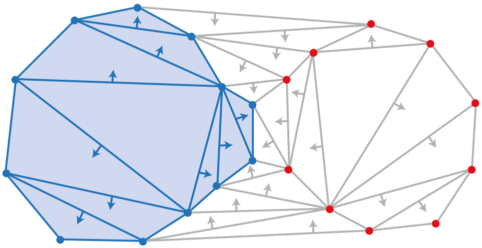
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



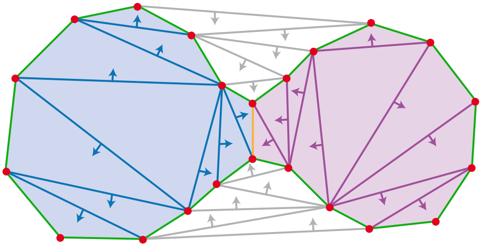
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



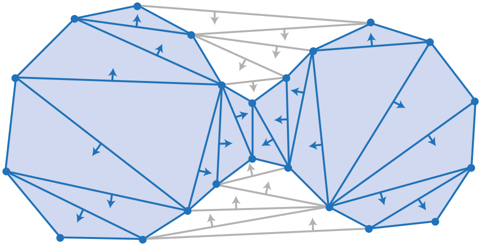
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Computing persistent homology via matrix reduction

Algorithm (matrix reduction; a variant of Gauss elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

$R = D$

$V = I(n)$

while there exist $i < j$ such that $\operatorname{pivot} R_i = \operatorname{pivot} R_j$ **do**

add column R_i to column R_j $\triangleright$ eliminate the nonzero entry in row $\operatorname{pivot} R_i$

add column V_i to column V_j

return R, V

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

```
function Reduce(D)
    R = D
    V = I(n)
    while there exist i < j such that R_j has a nonzero entry in row pivot R_i do
        add column R_i to column R_j          ▷ eliminate the nonzero entry in row pivot R_i
        add column V_i to column V_j
    return R, V
```

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

Require: D: $m \times n$ matrix
Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

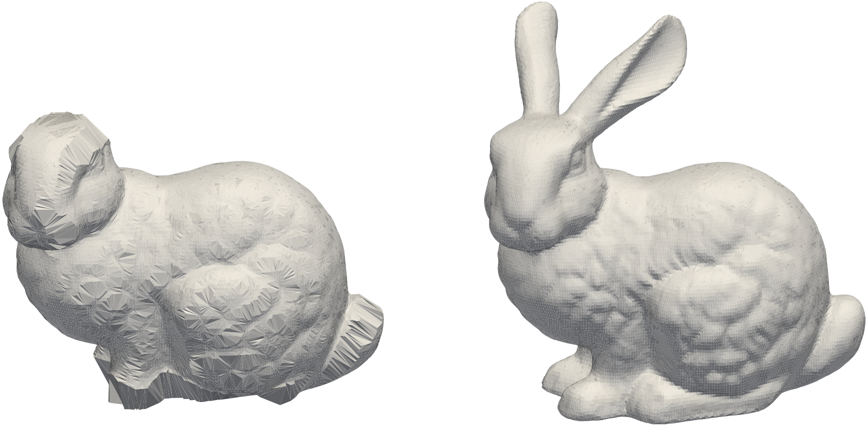
function Reduce(D)
  $R = D$
  $V = I(n)$
  while there exist $i < j$ such that R_j has a nonzero entry in row $\mathrm{pivot}\, R_i$ **do**
    add column R_i to column R_j ▷ eliminate the nonzero entry in row $\mathrm{pivot}\, R_i$
    add column V_i to column V_j
  return R, V

Proposition

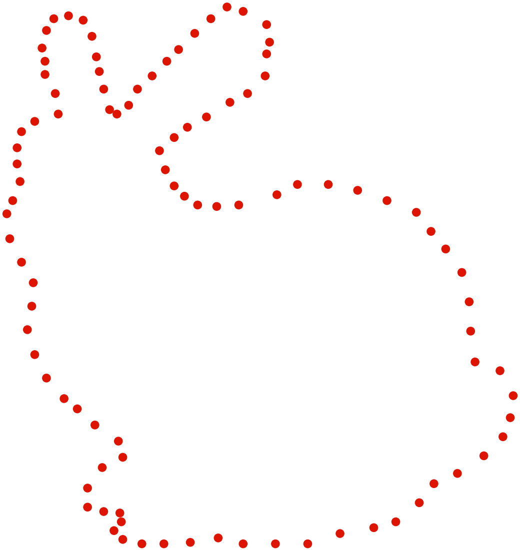
The resulting columns R_j are minimal (in a lexicographic order) within their homology class (in K_{j-1}).

Standard reduction and exhaustive reduction

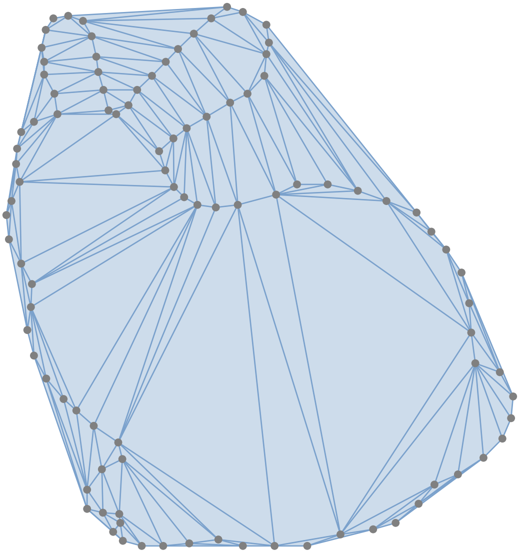
Standard reduction and exhaustive reduction



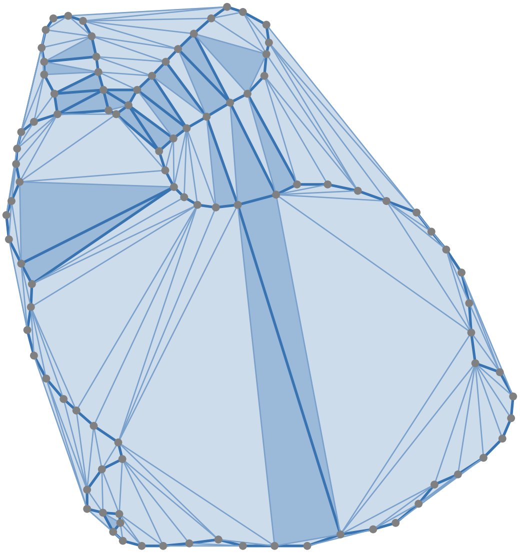
Wrap complexes and lexicographically minimal cycles



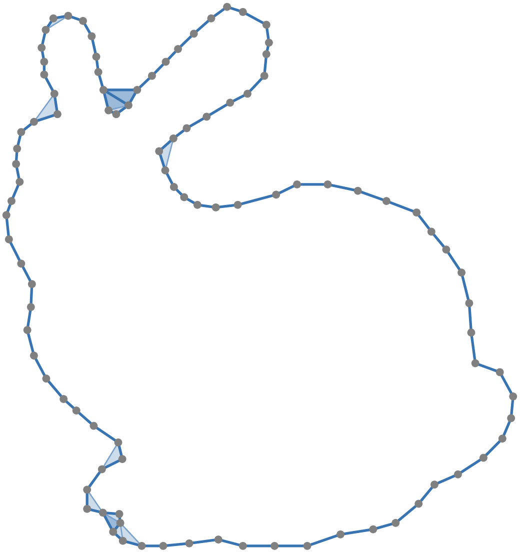
Wrap complexes and lexicographically minimal cycles



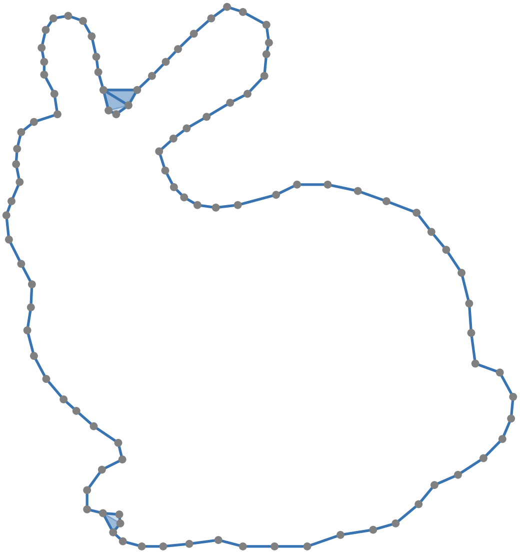
Wrap complexes and lexicographically minimal cycles



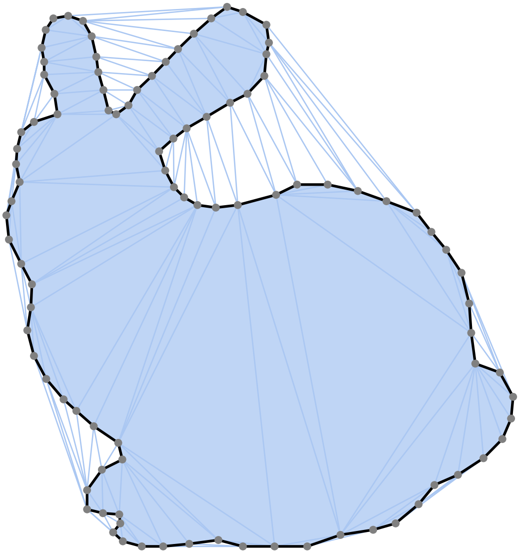
Wrap complexes and lexicographically minimal cycles



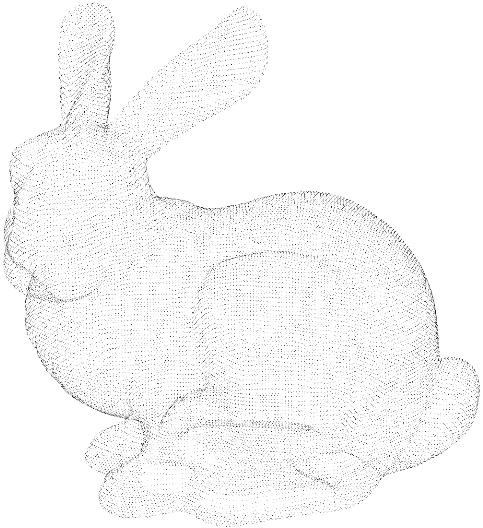
Wrap complexes and lexicographically minimal cycles



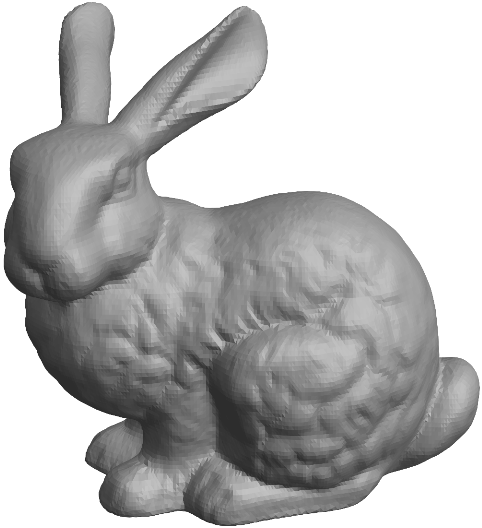
Wrap complexes and lexicographically minimal cycles



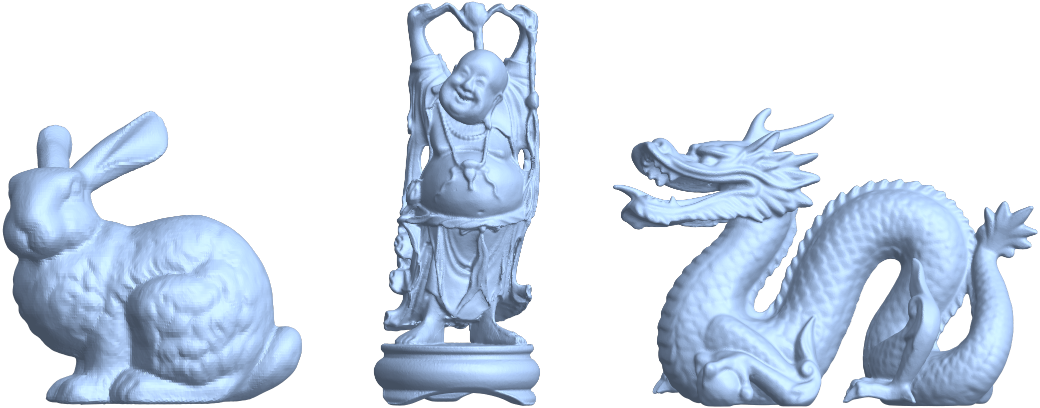
Wrap complexes and lexicographically minimal cycles



Wrap complexes and lexicographically minimal cycles



Point cloud reconstruction with minimal cycles

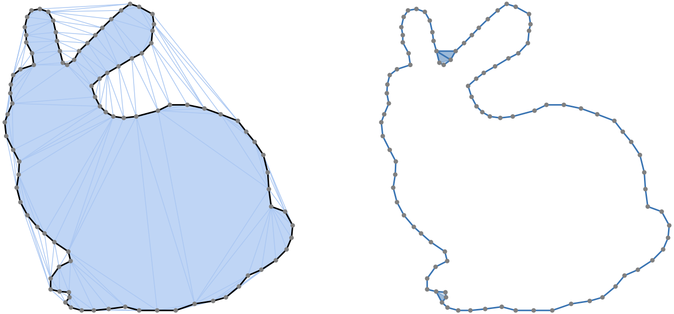


Wrap complexes support minimal cycles

Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- *Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.*
- *Any lexicographically minimal cycle of $\mathrm{Del}_r(X)$ is supported on $\mathrm{Wrap}_r(X)$.*

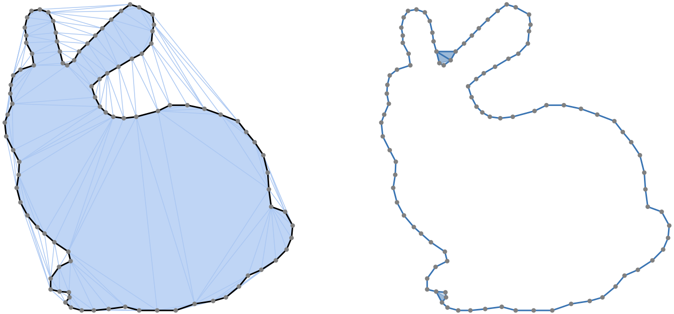


Wrap complexes support minimal cycles

Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- *Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.*
- *Any lexicographically minimal cycle of $\mathrm{Del}_r(X)$ is supported on $\mathrm{Wrap}_r(X)$.*



Apparent pairs form the bridge between persistent homology and discrete Morse theory

Thanks for your attention!

Thanks for your attention!

U. Bauer, F. Roll

Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

Symposium on Computational Geometry, 2022. doi:10.4230/LIPIcs.SoCG.2022.15

U. Bauer, F. Roll

Wrapping Cycles in Delaunay Complexes: Bridging Persistent Homology and Discrete Morse Theory

Symposium on Computational Geometry, 2024. arXiv:2212.02345

U. Bauer, H. Edelsbrunner

The Morse Theory of Čech and Delaunay Complexes

Transactions of the AMS, 2017. doi:10.1090/tran/6991

U. Bauer

Ripser: efficient computation of Vietoris–Rips persistence barcodes

Journal of Applied and Computational Topology , 2021. doi:10.1007/s41468-021-00071-5